

SECOND-ORDER HYPERBOLIC FUCHSIAN SYSTEMS. GOWDY SPACETIMES AND THE FUCHSIAN NUMERICAL ALGORITHM

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ABSTRACT. This is the second part of a series devoted to the singular initial value problem for second-order hyperbolic Fuchsian systems. In the first part, we defined and investigated this general class of systems, and we established a well-posedness theory in weighted Sobolev spaces. This theory is applied here to the vacuum *Einstein equations for Gowdy spacetimes* admitting, by definition, two Killing fields satisfying certain geometric conditions. We recover, by more direct and simpler arguments, the well-posedness results established earlier by Rendall and collaborators. In addition, in this paper we introduce a natural approximation scheme, which we refer to as the *Fuchsian numerical algorithm* and is directly motivated by our general theory. This algorithm provides highly accurate, numerical approximations of the solution to the singular initial value problem. In particular, for the class of Gowdy spacetimes under consideration, various numerical experiments are presented which show the interest and efficiency of the proposed method. Finally, as an application, we numerically construct Gowdy spacetimes containing a smooth, incomplete, non-compact Cauchy horizon.

1. INTRODUCTION

This is the second part of a series [3, 4] devoted to the initial value problem associated with the Einstein equations for spacetimes endowed with symmetries. We are typically interested in spacetimes with Gowdy symmetry, and in formulating the Einstein equations with data on a singular hypersurface, where curvature generically blows-up. In the first part [3], we introduced a class of partial differential equations, referred as *second-order Fuchsian systems*, and we established a general well-posedness theory within Sobolev spaces with weight (on the coordinate singularity). In the present paper, we tackle the treatment of actual models derived from the Einstein equations when suitable symmetry assumptions and gauge choices are made.

We consider here $(3+1)$ -dimensional, vacuum spacetimes (M, g) with spacelike slices diffeomorphic to the torus T^3 , satisfying the vacuum Einstein equations and the Gowdy symmetry assumption. That is, we assume the existence of an Abelian T^2 -isometry group with spacelike orbits and vanishing twist constants [8]. These so-called Gowdy spacetimes on T^3 were first studied in [13]. In the past years, a combination of theoretical and numerical works led to a detailed understanding

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of the Gowdy spacetimes, achieved by analyzing solutions to the Einstein equations as the singular boundary is approached; cf. [8, 15, 18, 20, 21, 22].

Before we can indicate precisely our contribution in the present paper, let us provide some background on Gowdy spacetimes. Introduce coordinates (t, x, y, z) such that (x, y, z) describe spatial sections diffeomorphic to T^3 while t is a timelike variable. We can arrange that the Killing fields associated with the Gowdy symmetry coincides with the coordinate vector fields ∂_y, ∂_z , in a global manner, so that the spacetime metric reads

$$(1.1) \quad g = \frac{1}{\sqrt{t}} e^{\Lambda/2} (-dt^2 + dx^2) + t (e^P (dy + Qdz)^2 + e^{-P} dz^2), \quad t > 0.$$

Hence, the metric depends upon three coefficients $P = P(t, x)$, $Q = Q(t, x)$, and $\Lambda = \Lambda(t, x)$. We also assume spatial periodicity with periodicity domain $U := [0, 2\pi)$.

In the chosen gauge, the Einstein's vacuum equations imply the following second-order wave equations for P, Q ,

$$(1.2) \quad \begin{aligned} P_{tt} + \frac{P_t}{t} - P_{xx} &= e^{2P} (Q_t^2 - Q_x^2), \\ Q_{tt} + \frac{Q_t}{t} - Q_{xx} &= -2(P_t Q_t - P_x Q_x), \end{aligned}$$

which are decoupled from the wave equation satisfied by the third coefficient Λ :

$$(1.3) \quad \Lambda_{tt} - \Lambda_{xx} = P_x^2 - P_t^2 + e^{2P} (Q_x^2 - Q_t^2).$$

Moreover, the Einstein equations also imply constraint equations, which read

$$(1.4a) \quad \Lambda_x = 2t (P_x P_t + e^{2P} Q_x Q_t),$$

$$(1.4b) \quad \Lambda_t = t (P_x^2 + t e^{2P} Q_x^2 + P_t^2 + e^{2P} Q_t^2).$$

It turns out that (1.3) can be ignored in the following sense. Given a time $t_0 > 0$, we can prescribe initial data $(P, Q)|_{t_0}$ for the system (1.2) while assuming the condition

$$(1.5) \quad \int_0^{2\pi} (P_x P_t + e^{2P} Q_x Q_t) dx = 0 \quad \text{at } t = t_0.$$

Then, the first constraint in (1.4) determines the function Λ at the initial time, up to a constant which we henceforth fix. Next, one easily checks that the solution (P, Q) of (1.2) corresponding to these initial data does satisfy the compatibility condition associated with (1.4) and, hence, (1.4) determines Λ uniquely for *all* times of the evolution. Moreover, one checks that (1.3) is satisfied identically by the constructed solution (P, Q, Λ) . Consequently, equations (1.2) represent the essential set of Einstein's field equations for Gowdy spacetimes. We refer to (1.2) as the *Gowdy equations* and focus our attention on them. One could also consider the alternative viewpoint which follows from the natural $3 + 1$ -splitting and treats the three equations (1.2)-(1.3) as an evolution system for the unknowns (P, Q, Λ) , and (1.4) will be regarded as constraints that propagate if they hold on an initial hypersurface.

Rendall and collaborators [2, 16, 19] developed the so-called *Fuchsian method* to handle singular evolution equations derived from the Einstein equations. This method allows one to derive precise information about the behavior of solutions near the singularity, which was a key step in the general proof of Penrose's strong cosmic conjecture eventually established by Ringström [22]. For the most recent developments we refer to [5, 6, 7]; especially, in [6], a generalization of the standard Fuchsian theory was recently introduced.

In the first part [3], we revisited the Fuchsian theory and developed another approach to the well-posedness theory in Sobolev spaces which applies to the singular initial value problem of a class of (second-order, hyperbolic) Fuchsian systems. The theory in [3] is particularly well-adapted to handle the Gowdy equations, as we show here. In fact, we propose here a rather simple proof of the well-posedness of the singular initial value problem for the Gowdy equations, which provides an alternative to the approach in Rendall [19]. Moreover, in passing, we are also able to clarify certain issues.

A second objective in the present paper is to present a new numerical approach and apply it specifically to Gowdy spacetimes. This approach is inspired by a pioneering work by Amorim, Bernardi, and LeFloch [1], which computed solutions to the Gowdy equations with data imposed on the singularity. The approximation scheme proposed in the present paper, which we refer to as the *Fuchsian numerical algorithm*, is directly derived from our existence theory [3] and relies strongly on the hyperbolicity of the equations, so that error estimates for the numerical approximations are expected to hold.

This paper is organized as follows. Section 2 is devoted to the theoretical discussion of the singular initial value problem for the Gowdy equations. First, we recall some heuristics for the behavior of the solutions at the “singularity” $t = 0$. Then, we show that our notion of singular initial value problems in [3] is consistent with these heuristics, and then state the well-posedness results that follows from the first part. In Section 4, we introduce a general numerical approach to solve second-order hyperbolic Fuchsian systems numerically and then apply it to the Gowdy equations. Several numerical experiments are presented and, finally we numerically construct a Gowdy symmetric solution of Einstein’s field equations with a smooth non-compact Cauchy horizon.

2. SINGULAR INITIAL VALUE PROBLEM FOR THE GOWDY EQUATIONS

2.1. Heuristics about the Gowdy equations. We provide here a formal discussion which motivates the following (rigorous) analysis. Based on extensive numerical experiments, it was first conjectured (and later established rigorously) that as one approaches the singularity the spatial derivative of solutions (P, Q) to (1.2) become negligible and (P, Q) , should approach a solution of the *ordinary* differential equations

$$(2.1) \quad \begin{aligned} P_{tt} + \frac{P_t}{t} &= e^{2P} Q_t^2, \\ Q_{tt} + \frac{Q_t}{t} &= -2 P_t Q_t. \end{aligned}$$

These equations are referred to, in the literature, as the *velocity term dominated* (VTD) equations. Interestingly enough, they admit a large class of solutions given explicitly by

$$(2.2) \quad \begin{aligned} P(t, x) &= \ln(\alpha t^k (1 + \zeta^2 t^{-2k})), \\ Q(t, x) &= \xi - \frac{\zeta t^{-2k}}{\alpha (1 + \zeta^2 t^{-2k})}, \end{aligned}$$

where x plays simply the role of a parameter and α, ζ, ξ, k are arbitrary 2π -periodic functions of x .

Based on (2.2), it is a simple matter to determine the first two terms in the expansion of the function P near $t = 0$, that is, for $k \neq 0$ at least,

$$\lim_{t \rightarrow 0} \frac{P(t, x)}{\ln t} = \lim_{t \rightarrow 0} t P_t(t, x) = -|k|,$$

hence

$$\lim_{t \rightarrow 0} (P(t, x) + |k(x)| \ln t) = \varphi(x), \quad \varphi := \begin{cases} \ln \alpha, & k < 0, \\ \ln(\alpha(1 + \zeta^2)), & k = 0, \\ \ln(\alpha\zeta^2), & k > 0. \end{cases}$$

Similarly, for the function Q we obtain

$$\lim_{t \rightarrow 0} Q(t, x) = q(x), \quad q := \begin{cases} \xi, & k < 0, \\ \xi - \frac{\zeta}{\alpha(1 + \zeta^2)}, & k = 0, \\ \xi - \frac{1}{\alpha\zeta}, & k > 0, \end{cases}$$

$$\lim_{t \rightarrow 0} t^{-2|k|} (Q(t, x) - q(x)) = \psi(x), \quad \psi := \begin{cases} -\frac{\zeta}{\alpha}, & k < 0, \\ 0, & k = 0, \\ \frac{1}{\alpha\zeta^3}, & k > 0. \end{cases}$$

From (2.2), we thus have the expansion

$$(2.3) \quad \begin{aligned} P &= -|k| \ln t + \varphi + o(1), \\ Q &= q + t^{2|k|} \psi + o(t^{2|k|}), \end{aligned}$$

in which k, φ, q, ψ are functions of x . In general, P blows-up to $+\infty$ when one approaches the singularity, while Q remains *bounded*. Observe that the sign of k is irrelevant as far the asymptotic expansion is concerned, and we are allowed to restrict attention to $k \geq 0$.

By plugging the explicit solution in the nonlinear terms arising in (2.1) one sees that $e^{2P} Q_t^2$ is of order $t^{2(|k|-1)}$ which is negligible since the left-hand side of the P -equation is of order t^{-2} , at least when $k \neq 0$. On the other hand, the nonlinear term $P_t Q_t$ is of order $t^{2(|k|-1)}$, which is the same order as the left-hand side of the Q -equation. It is not negligible, but we observe that $P_t Q_t$ has the same behavior as $-(|k|/t) Q_t$.

In fact, observe that the homogeneous system deduced from (2.1):

$$(2.4) \quad P_{tt} + \frac{P_t}{t} = 0, \quad Q_{tt} + \frac{1-2k}{t} Q_t = 0,$$

is solved precisely by the leading-order terms in (2.3). This tells us that, as $t \rightarrow 0$, the term $e^{2P} Q_t^2$ is negligible in the first equation in (2.1), while $P_t Q_t + (|k|/t) Q_t$ is negligible at $t = 0$. This discussion hence allows us to conclude that as far as the behavior at the coordinate singularity $t = 0$ is concerned, the nonlinear VTD equations (2.1) are well approximated by the system (2.4).

We return now to the nonlinear terms which were not included in the VTD equations, but yet are present in the full model (1.2). Allowing ourselves to differentiate the expansion (2.3), we get the following leading-order terms at $t = 0$:

$$e^{2P} Q_x^2 = \begin{cases} t^{-2|k|} e^{2\varphi} q_x^2 + \dots, & q_x \neq 0, \\ 2e^{2\varphi} |k|_x \psi \ln t + \dots, & q_x = 0, \quad |k|_x \neq 0, \\ e^{2\varphi} \psi_x^2 + \dots, & q_x = 0, \quad |k|_x = 0, \quad \psi_x \neq 0, \end{cases}$$

$$P_x Q_x = \begin{cases} -\ln t |k|_x q_x + \dots, & |k|_x, q_x \neq 0, \\ \varphi_x q_x + \dots, & |k|_x = 0, \quad \varphi_x, q_x \neq 0, \\ -2 \ln^2 t t^{2|k|} (|k|_x)^2 \psi + \dots, & |k|_x \neq 0, \quad q_x = 0 \\ t^{2|k|} \varphi_x \psi_x + \dots, & |k|_x = q_x = 0, \quad \varphi_x, \psi_x \neq 0. \end{cases}$$

To check (formally) the validity of the expansion (2.3) we now return to the full system. Consider the nonlinear term $e^{2P} Q_x^2$ in (1.2). and observe the following:

- Case $q_x \neq 0$ everywhere on an open subinterval of $[0, 2\pi]$. Then, on one hand, the left-hand side of the first equation in (1.2) is of order t^{-2} , at most. On the other hand, the term $e^{2P} Q_x^2$ is negligible with respect to t^{-2} if and only if the asymptotic velocity satisfies $|k| < 1$ and is of the same order if $|k| = 1$.
- Case $q_x = 0$ on an open subinterval of $[0, 2\pi]$. Then, $e^{2P} Q_x^2$ is negligible with respect to t^{-2} , and no condition on the velocity k is required on that interval.
- Case $q_x(x_0) = 0$ at some isolated point x_0 . Then, no definite conclusion can be obtained and a “competition” between $|k|$ (which may approach the interval $[0, 1]$) and q_x (which approaches zero) is expected.

Similarly, at least when $|k|_x q_x \neq 0$, the nonlinear term $P_x Q_x$ is of order $\ln t$ and, therefore, negligible with respect to $t^{2(|k|-1)}$ (given by the left-hand side of the second equation in (1.2)) if and only if the asymptotic velocity is $|k| \leq 1$. Points where $|k|_x$ or q_x vanish lead to a less singular behavior and condition on the velocity can also be relaxed.

The formal derivation above strongly suggests that we seek solutions to the full nonlinear equations admitting an asymptotic expansion of the form (2.3), that is

$$P = -k \ln t + \varphi + o(1), \quad Q = q + t^{2k} (\psi + o(1)),$$

where $k \geq 0$ and φ, q, ψ are prescribed. In other words, these solutions asymptotically approach a solution of the VTD equations and, in consequence, such solutions will be referred to as *asymptotically velocity term dominated* (AVTD) solutions.

Based on this analysis and extensive numerical experiments, it has been conjectured that asymptotically as one approaches the coordinate singularity $t = 0$ the function $P(t, x)/\ln t$ should approach some limit $k = k(x)$, referred to as the *asymptotic velocity*, and that $k(x)$ should belong to $[0, 1]$ with the exception of a zero measure set of “exceptional values”.

2.2. Gowdy equations as a second-order hyperbolic Fuchsian system. The first step in our (rigorous) analysis of the Gowdy equations (1.2) now is to write them as a system of second-order hyperbolic Fuchsian equations. These are equations of the form (written here for a scalar field in order to keep the notation simple)

$$(2.5) \quad D^2 v(t, x) + 2a(x) Dv(t, x) + b(x)v(t, x) = t^2 c^2(t, x) \partial_x^2 v(t, x) + f(t, x, v, Dv, \partial_x v),$$

where $v : (0, \delta] \times U \rightarrow \mathbb{R}$ is the unknown defined on an interval $U \subset \mathbb{R}$ an interval (with $\delta > 0$). Here we use the symbol $D := t\partial_t$, which denotes the time derivative operator and is singular at the origin $t = 0$; by $D^2 v$ we mean $D(Dv) = t\partial_t(t\partial_t v)$. We assume that all quantities are periodic in x with a periodicity domain U , and for the present applications we set $U := [0, 2\pi]$. The coefficients a and b are smooth and depend on the spatial coordinate x , only. The characteristic speed c has to satisfy certain properties at $t = 0$; however, since in our application we have $c \equiv 1$, we do not need to discuss those here. All further details are explained in [3]. The left-hand side of the equation is called the *principal part*, and its right-hand side the *source-term*. In the course of the discussion, we will often abuse notation slightly and refer to the function f alone as the source-term.

In the first part [3], we determined leading-order behavior of solutions to (2.5) at $t = 0$, from the assumptions that it is “driven” by the principal part of the equation (in a well-defined manner) and, additionally, the source-term has a suitable decay property at $t = 0$. In particular, spatial derivatives are negligible, in a sense made precise in [3].

From the equation (2.5), we define

$$u_0(t, x) := \begin{cases} u_*(x) t^{-a(x)} \ln t + u_{**}(x) t^{-a(x)} & a^2 = b, \\ u_*(x) t^{-\lambda_1(x)} + u_{**}(x) t^{-\lambda_2(x)}, & a^2 \neq b, \end{cases}$$

and

$$(2.6) \quad \lambda_1 := a + \sqrt{a^2 - b}, \quad \lambda_2 := a - \sqrt{a^2 - b}.$$

Now, u_0 is regarded a prescribed (real-valued) function and, as far as our application to (1.2) is concerned, λ_1 and λ_2 are real-valued. The function u_0 is smooth on $(0, \delta] \times U$, provided either $a^2(x) \neq b(x)$ for all $x \in U$ or else $a^2(x) = b(x)$ for all $x \in U$. More generally, when there is a transition between these two regimes, the functions u_* and u_{**} have to be renormalized [3] in order to guarantee the smoothness of u_0 , which we assume from now on. The function u_0 represents the leading-order terms in the “*singular initial value problem with two-term asymptotic data u_* and u_{**}* ”, as discussed in [3] and is referred to as a *canonical two-term expansion*.

After multiplication by t^2 , the equations (1.2) immediately take the second-order hyperbolic Fuchsian form

$$\begin{aligned} D^2 P &= t^2 \partial_x^2 P + e^{2P} (DQ)^2 - t^2 e^{2P} (\partial_x Q)^2, \\ D^2 Q &= t^2 \partial_x^2 Q - 2DPDQ + 2t^2 \partial_x P \partial_x Q. \end{aligned}$$

The general canonical two-term expansion then reads

$$P(t, x) = P_*(x) \log t + P_{**}(x) + \dots$$

for the function P and, similarly, an expansion $Q_*(x) \log t + Q_{**}(x) + \dots$ for the function Q with prescribed data Q_*, Q_{**} . At this stage, we do not make precise statements about the (higher-order) remainders, yet. In any case, the theory in [3] does not apply to this system directly, due to the presence of the term $-2DPDQ$ —with the exception of the cases $P_* = 0$ or $Q_* = 0$. Namely, this term does not behave as a positive power of t at $t = 0$ when we substitute P and Q by their canonical two-term expansions, but this is required by the theory.

At this juncture, motivated by the formal discussion in Section 2.1, especially (2.4), we propose to add a term $-2kDQ$ to the equation for Q where k is a prescribed (smooth, spatially periodic) function depending on x , only. This yields the system of equations

$$(2.7) \quad \begin{aligned} D^2 P &= t^2 \partial_x^2 P + e^{2P} (DQ)^2 - t^2 e^{2P} (\partial_x Q)^2, \\ D^2 Q - 2kDQ &= t^2 \partial_x^2 Q - 2(k + DP)DQ + 2t^2 \partial_x P \partial_x Q. \end{aligned}$$

Later, the function k will play the role of the asymptotic velocity mentioned before. The resulting system is of second-order hyperbolic Fuchsian form with two equations, corresponding to

$$(2.8) \quad \lambda_1^{(1)} = \lambda_2^{(1)} = 0, \quad \lambda_1^{(2)} = 0, \quad \lambda_2^{(2)} = -2k.$$

Here, the superscript determines the respective equation of the system (2.7). If we assume that k is a strictly positive function, as we will do in all what follows, the expected leading-order behavior

at $t = 0$ given by the canonical two-term expansions is

$$(2.9) \quad \begin{aligned} P(t, x) &= P_*(x) \log t + P_{**}(x) + \dots, \\ Q(t, x) &= Q_*(x) + Q_{**}(x) t^{2k(x)} + \dots \end{aligned}$$

One checks easily that the problem, which we had for the previous form of the equations, does not arise if $P_* = -k$. Indeed, the canonical two-term expansion is consistent with the heuristics of the Gowdy equations above and we recover the singular initial value problem studied rigorously in [16, 19] and numerically in [1]. We only mention here without further notice that the case $k = 0$ with the logarithmic canonical two-term expansion for Q given by the first form of the equations above is covered by the following discussion. Furthermore, the case of possibly vanishing k may be also included via a suitable normalization of the asymptotic data [3].

3. WELL-POSEDNESS THEORY FOR THE GOWDY EQUATIONS

3.1. Reformulation of the problem. When $P_* = -k$, this function plays a two-fold role in (2.7). On one hand, it is an asymptotic data for the function P and, on the other hand, it is a coefficient of the equation satisfied by the function Q . In order to keep these two roles of k separated in a first stage and instead of (2.7), we consider the system

$$(3.1) \quad \begin{aligned} D^2 P &= t^2 \partial_x^2 P + e^{2P} (DQ)^2 - t^2 e^{2P} (\partial_x Q)^2, \\ D^2 Q - 2k DQ &= t^2 \partial_x^2 Q - 2(-P_* + DP) DQ + 2t^2 \partial_x P \partial_x Q. \end{aligned}$$

Studying the singular initial value problem with two-term asymptotic data means that we search for solutions to (3.1) of the form (as $t \rightarrow 0$)

$$(3.2) \quad \begin{aligned} P(t, x) &= P_*(x) \log t + P_{**}(x) + w^{(1)}(t, x), \\ Q(t, x) &= Q_*(x) + Q_{**}(x) t^{2k(x)} + w^{(2)}(t, x), \end{aligned}$$

for general asymptotic data P_* , P_{**} , Q_* , Q_{**} , and remainders $w^{(1)}$, $w^{(2)}$ which will be specified to be suitably “small” and belong to certain functional spaces; see below. After studying the well-posedness for this problem, we can always choose P_* to coincide with $-k$ and, therefore, recover our original Gowdy problem (2.7)-(2.9). For simplicity in the presentation, we always assume that k is a C^∞ function.

In the following discussion, we write the vector-valued remainder as $w := (w^{(1)}, w^{(2)})$, and we fix some asymptotic data P_* , P_{**} , Q_* , and Q_{**} . In agreement with the notation in [3], the source-term operator $F = (F_1, F_2)$ is defined by

$$F[w](t, x) := f\left(t, x, u_0 + w, D(u_0 + w), \partial_x(u_0 + w)\right),$$

where u_0 is the vector-valued, canonical two-term expansion given by (3.2) and the asymptotic data. We write

$$F[w](t, x) =: (F_1[w](t, x), F_2[w](t, x)),$$

and from (3.1) we obtain

$$\begin{aligned} F_1[w] &= \left(t^{P_*} e^{P_{**}} e^{w^{(1)}} (2k t^{2k} Q_{**} + D w^{(2)}) \right)^2 \\ &\quad - \left(t^{P_*} e^{P_{**}} e^{w^{(1)}} (t \partial_x Q_* + 2 \partial_x k t^{2k} t \ln t Q_{**} + t^{2k} t \partial_x Q_{**} + t \partial_x w^{(2)}) \right)^2 \end{aligned}$$

and

$$\begin{aligned} F_2[w] &= -2Dw^{(1)} (2kt^{2k}Q_{**} + Dw^{(2)}) \\ &\quad + 2\left(t\partial_x P_* \log t + t\partial_x P_{**} + t\partial_x w^{(1)}\right) \left((t\partial_x Q_* + 2\partial_x kt^{2k}t \ln t Q_{**} + t^{2k}t\partial_x Q_{**} + t\partial_x w^{(2)})\right). \end{aligned}$$

3.2. Properties of the source-term operator. To establish the well-posedness of the singular initial value problem for the Gowdy equations, we need first to derive certain decay properties of the source-term operator F . In [3], we introduced the spaces $X_{\delta,\alpha,k}$ and $\tilde{X}_{\delta,\alpha,k}$ associated with constants $\delta, \alpha > 0$ and non-negative integers k : in short, a function w belongs to $X_{\delta,\alpha,k}$ if each derivative $\partial_x^l D^m w$ (with $l + m \leq k$) weighted by $t^{\lambda_2(x)-\alpha}$ is a bounded continuous map $(0, \delta] \rightarrow L^2(U)$. Here, λ_2 given by (2.6) is determined by the coefficients of the second-order hyperbolic Fuchsian equation under consideration. The corresponding norm $\|\cdot\|_{\delta,\alpha,k}$ turns $(X_{\delta,\alpha,k}, \|\cdot\|_{\delta,\alpha,k})$ into a Banach space. For any $w \in X_{\delta,\alpha,k}$, this norm has the form $\|w\|_{\delta,\alpha,k} = \sup_{t \in (0, \delta]} E_{\delta,\alpha,k}[w](t)$, where $E_{\delta,\alpha,k}[w] : (0, \delta] \rightarrow \mathbb{R}$ is bounded and continuous.

The spaces $\tilde{X}_{\delta,\alpha,k}$ and the associated maps $\tilde{E}_{\delta,\alpha,k}$ are defined analogously; only the weight of the k -th spatial derivative is substituted by¹ $t^{\lambda_2(x)-\alpha+1}$. It always follows that $X_{\delta,\alpha,k} \subset \tilde{X}_{\delta,\alpha,k}$. There is, in fact, no reason to assume $\alpha > 0$ to remain constant in the definition of these spaces, since no essentially new difficulty arises in the theory [3] when α is a (spatially periodic) strictly positive function in $C^1(U)$.

Let us introduce some further notation specific to the Gowdy equations. Let $X_{\delta,\alpha_1,k}^{(1)}$ be the space defined as above and based on the coefficients of the first equation in (3.1) and, similarly, let $X_{\delta,\alpha_2,k}^{(2)}$ be the space associated with the second equation. By definition, a vector-valued map $w = (w^{(1)}, w^{(2)})$ belongs $X_{\delta,\alpha,k}$ precisely if $w^{(1)} \in X_{\delta,\alpha_1,k}^{(1)}$ and $w^{(2)} \in X_{\delta,\alpha_2,k}^{(2)}$, with $\alpha := (\alpha_1, \alpha_2)$. An analogous notation is used for the spaces $\tilde{X}_{\delta,\alpha_1,k}^{(1)}$, $\tilde{X}_{\delta,\alpha_2,k}^{(2)}$ and $\tilde{X}_{\delta,\alpha,k}$.

Now we are ready to state a first result about the source-term of (3.1).

Lemma 3.1 (Operator F in the finite differentiability class). *Fix any $\delta > 0$ and any asymptotic data*

$$P_*, P_{**}, Q_*, Q_{**} \in H^m(U) \quad m \geq 2.$$

Suppose there exist $\epsilon > 0$ and a continuous function $\alpha = (\alpha_1, \alpha_2) : U \rightarrow (0, \infty)^2$ so that, at each $x \in U$,

$$(3.3a) \quad \alpha_1(x) + \epsilon < \min(2(P_*(x) + 2k(x)), 2(P_*(x) + 1)),$$

$$(3.3b) \quad \alpha_2(x) + \epsilon < 2(1 - k(x)),$$

$$(3.3c) \quad \alpha_1(x) - \alpha_2(x) > \epsilon + \min(0, 2k(x) - 1),$$

$$(3.3d) \quad \epsilon < 1.$$

Then, the operator F associated with the system (3.1) and the given asymptotic data maps $\tilde{X}_{\delta,\alpha,m}$ into $X_{\delta,\alpha+\epsilon,m-1}$ and satisfies the following Lipschitz continuity condition: For each $r > 0$ and for some constant $C > 0$ (independent of δ),

$$E_{\delta,\alpha+\epsilon,m-1}[F[w] - F[\hat{w}]](t) \leq C \tilde{E}_{\delta,\alpha,m}[w - \hat{w}](t), \quad t \in (0, \delta]$$

¹The (slightly) more general definition given in [3] involved a speed coefficient c , taken here to be identically unit.

for all $w, \hat{w} \in B_r \subset \tilde{X}_{\delta, \tilde{\alpha}, m}$, where B_r denotes the closed ball centered at the origin.

In this lemma, since $P_* \in H^1(U)$, in particular, a standard Sobolev inequality implies that P_* can be identified with a unique bounded continuous periodic function on U , and the inequality (3.3a) makes sense pointwise.

Proof. Consider the expression of F given at the end of Section 3.1. Let $w \in \tilde{X}_{\delta, \alpha, m}$ for some (so far unspecified) positive spatially dependent functions α_1, α_2 , hence $w^{(1)} \in \tilde{X}_{\delta, \alpha_1, m}^{(1)}$ and $w^{(2)} \in \tilde{X}_{\delta, \alpha_2, m}^{(2)}$. By a standard Sobolev inequality (since $m \geq 2$ and the spatial dimension is 1), we get that $F[w](t, \cdot) \in H^{m-1}(U)$ for all $t \in (0, \delta]$. Namely, if $m \geq 2$ we can control the non-linear terms of $F[w](t, \cdot)$ in all generality for a given $t > 0$ if any factor in any term of $F[w](t, \cdot)$, after applying up to $m-1$ spatial derivatives, is an element in $L^\infty(U)$ – with the exception of the m th spatial derivative of w which is only required to be in $L^2(U)$. This is guaranteed by the Sobolev inequalities. Having found that $F[w](t, \cdot) \in H^{m-1}(U)$ for all $t \in (0, \delta]$, it is easy to check that $F_1[w] \in X_{\delta, \alpha_1 + \epsilon, 0}^{(1)}$ if

$$(3.4) \quad \alpha_1(x) + \epsilon \leq \min \left(2(P_*(x) + 2k(x)), 2(P_*(x) + 1) \right), \quad x \in U.$$

Even more, condition (3.4) implies that $D^l F_1[w] \in X_{\delta, \alpha_1 + \epsilon, 0}^{(1)}$ for all $l \leq m-1$.

Considering now spatial derivatives, we have to deal with two difficulties. The first one is that logarithmic terms arise with each spatial derivative. We find $\partial_x^k D^l F_1[w] \in X_{\delta, \alpha_1 + \epsilon, 0}^{(1)}$ for all $l \leq m-1$ and $k \leq m-2$ and $k+l \leq m-1$ (excluding first the case $k = m-1, l = 0$) provided

$$(3.5) \quad \alpha_1(x) + \epsilon < \min \left(2(P_*(x) + 2k(x)), 2(P_*(x) + 1) \right), \quad x \in U.$$

A second difficulty arises in the case $k = m-1, l = 0$. Namely, since $w \in \tilde{X}_{\delta, \alpha, m}$ (and not in $X_{\delta, \alpha, m}$), it follows that in particular $t \partial_x^m w^{(2)} \sim t^{2k+\alpha_2}$ (and not $t^{1+2k+\alpha_2}$); note that the function β which determines the behavior of the characteristic speeds at $t = 0$ in [3] is identically zero in the case of the Gowdy equations. The potentially problematic term is hence of the form AB with

$$\begin{aligned} A &:= t^{P_*} e^{P_{**}} e^{w^{(1)}} (t \partial_x Q_* + 2 \partial_x k t^{2k} t \ln t Q_{**} + t^{2k} t \partial_x Q_{**} + t \partial_x w^{(2)}), \\ B &:= \cdot t^{P_*} e^{P_{**}} e^{w^{(1)}} (\partial_x^{m-1} (t \partial_x Q_* + 2 \partial_x k t^{2k} t \ln t Q_{**} + t^{2k} t \partial_x Q_{**}) + t \partial_x^m w^{(2)}), \end{aligned}$$

originating from taking $m-1$ spatial derivatives of $F_1[w]$. To ensure $\partial_x^{m-1} F_1[w] \in X_{\delta, \alpha_1 + \epsilon, 0}^{(1)}$, we need

$$(3.6) \quad \alpha_1(x) + \epsilon < (P_*(x) + 1) + (P_*(x) + 2k(x) + \alpha_2(x)), \quad x \in U.$$

If (3.5) is satisfied, we have (for all x)

$$\begin{aligned} \alpha_1(x) + \epsilon &< \min \left(2(P_*(x) + 2k(x)), 2(P_*(x) + 1) \right) \\ &\leq (P_*(x) + 1) + (P_*(x) + 2k(x)) \end{aligned}$$

and, thus, (3.6) follows from (3.5). In conclusion, (3.5) is sufficient to guarantee that $F_1[w] \in X_{\delta, \alpha_1 + \epsilon, m-1}^{(1)}$.

Let us proceed next with the analysis of the term $F_2[w]$. If

$$(3.7) \quad \alpha_1(x) - \alpha_2(x) \geq \epsilon, \quad \alpha_2(x) + \epsilon < 2(1 - k(x)), \quad x \in U,$$

then $F_2[w] \in X_{\delta, \alpha_2 + \epsilon, 0}^{(2)}$. This inequality also implies that all time derivatives are in $X_{\delta, \alpha_2 + \epsilon, 0}^{(2)}$ as before. We have to deal with the same two difficulties as before when we consider spatial derivatives

of $F_2[w]$. On one hand, equality in (3.7) cannot occur due to additional logarithmic terms. On the other hand, we must be careful with the $(m-1)$ -th spatial derivative of $F_2[w]$. Here, the two problematic terms are of the form AB with either

$$\begin{aligned} A &:= \partial_x^{m-1}(t\partial_x P_* \log t + t\partial_x P_{**}) + t\partial_x^m w^{(1)}, \\ B &:= t\partial_x Q_* + 2\partial_x k t^{2k} t \ln t Q_{**} + t^{2k} t\partial_x Q_{**} + t\partial_x w^{(2)}, \end{aligned}$$

or else

$$\begin{aligned} A &:= t\partial_x P_* \log t + t\partial_x P_{**} + t\partial_x w^{(1)}, \\ B &:= \partial_x^{m-1}(t\partial_x Q_* + 2\partial_x k t^{2k} t \ln t Q_{**} + t^{2k} t\partial_x Q_{**}) + t\partial_x^m w^{(2)}. \end{aligned}$$

The first one is under control provided $\alpha_1(x) + 1 > 2k(x) + \alpha_2(x) + \epsilon$, for all $x \in U$, while for the second one it is sufficient to require $\epsilon < 1$. The claimed Lipschitz continuity condition follows from the above arguments. \square

Obviously, positive functions α_1 and α_2 and constants $\epsilon > 0$ satisfying the hypothesis of Lemma 3.1 can exist only if $k(x) < 1$ for all $x \in U$ (due to (3.3b)) except for special choices of data; cf. Lemma 3.3, below. Hence, we make the assumption that $0 < k(x) < 1$ for all x , which is consistent with our formal analysis in Section 2.1. As a consistency check for the case of interest $P_* = -k$, let us determine under which conditions the inequalities (3.3) can be hoped to be satisfied at all. For this, consider (3.3a) and (3.3c) in the “extreme” case $\alpha_2 = \epsilon = 0$. This leads to the condition $0 < k < 3/4$, which shows that Lemma 3.1 does not apply within the full interval $0 < k < 1$.

It is interesting to note that Rendall was led to the same restriction in [19], but its origin was not obvious in his approach. Here, we find that this is caused by the presence of the condition (3.3c) in particular which reflects the fact that w is an element of the space $\tilde{X}_{\delta,\alpha,m}$ rather than of the smaller space $X_{\delta,\alpha,m}$. Interestingly, we can eliminate this condition and, hence, retain the full interval $0 < k < 1$, when we consider the C^∞ -case, instead of finite differentiability, as we now show.

Lemma 3.2 (Operator F in the C^∞ class. General theory). *Fix any $\delta > 0$ and any asymptotic data*

$$P_*, P_{**}, Q_*, Q_{**} \in C^\infty(U).$$

Suppose there exist a constant $\epsilon > 0$ and a continuous functions $\alpha = (\alpha_1, \alpha_2) : U \rightarrow (0, \infty)^2$ such that, at each $x \in U$,

$$(3.8a) \quad \alpha_1(x) + \epsilon < \min(2(P_*(x) + 2k(x)), 2(P_*(x) + 1)),$$

$$(3.8b) \quad \alpha_2(x) + \epsilon < 2(1 - k(x)),$$

$$(3.8c) \quad \alpha_1(x) - \alpha_2(x) > \epsilon.$$

Then, for each integer $m \geq 1$, the operator F maps $X_{\delta,\alpha,m}$ into $X_{\delta,\alpha+\epsilon,m-1}$ and satisfies the following Lipschitz continuity property: for each $r > 0$ and some constant $C > 0$ (independent of δ),

$$E_{\delta,\alpha+\epsilon,m-1}[F[w] - F[\hat{w}]](t) \leq C\tilde{E}_{\delta,\alpha,m}[w - \hat{w}](t), \quad t \in (0, \delta],$$

for all $w, \hat{w} \in B_r \cap X_{\delta,\alpha+\epsilon,m} \subset \tilde{X}_{\delta,\alpha+\epsilon,m}$.

The proof is completely analogous to that of Lemma 3.1. Recall from [3] that only spaces without the tilde are necessary for the well-posedness theory in the C^∞ -case and hence that we obtain stronger control than in the finite differentiability case. Hence, the C^∞ -case does not require the condition (3.3c). This has the consequence that k can have values in the whole interval $(0, 1)$ as we show in detail later. In a special case, which will be of interest for the later discussion, however, we can relax the constraints for k even in the finite differentiability case.

Lemma 3.3 (Operator F in the finite differentiability class. A special case). *Fix any $\delta > 0$ and any asymptotic data*

$$P_*, P_{**}, Q_{**} \in H^m(U), \quad Q_* = \text{const}, \quad m \geq 2.$$

Suppose there exist $\epsilon > 0$ and a continuous function $\alpha = (\alpha_1, \alpha_2) : U \rightarrow (0, \infty)^2$ such that, at each $x \in U$,

$$\begin{aligned} \alpha_1(x) + \epsilon &< 2(P_*(x) + 2k(x)), \\ \alpha_2(x) + \epsilon &< 2, \\ \alpha_1(x) - \alpha_2(x) &> \epsilon - 1, \\ \epsilon &< 1. \end{aligned}$$

Then, the operator F satisfies the conclusions of Lemma 3.1.

In the special case of constant asymptotic data $Q_* = \text{const}$, we can prove the required properties of F if the function k is any positive function in the finite differentiability case. The analogous result for the C^∞ -case can also be derived.

3.3. Well-posedness of the singular initial value problem. Relying on Theorem 3.10 in [3], we now determine conditions that ensure that the singular initial value problem for the Gowdy equations is well-posed. Besides the properties of the source-operator F already discussed, we have to check the positivity of the *energy dissipation matrix*

$$N := \begin{pmatrix} \Re(\lambda_1 - \lambda_2) + \alpha & ((\Im \lambda_1)^2 / \eta - \eta) / 2 & 0 \\ ((\Im \lambda_1)^2 / \eta - \eta) / 2 & \alpha & t \partial_x c - \partial_x \Re(\lambda_1 - \lambda_2)(tc \ln t) \\ 0 & t \partial_x c - \partial_x \Re(\lambda_1 - \lambda_2)(tc \ln t) & \Re(\lambda_1 - \lambda_2) + \alpha - 1 - Dc/c \end{pmatrix}$$

for some well-chosen constant $\eta > 0$ and for each of the two Gowdy equations. Here, we omit the upper indices order to simplify the notation, while \Re and \Im denote the real and imaginary part of a complex number. In the present paper, the characteristic speed c in [3] is constant equal to 1, and all eigenvalues are real, so that the above matrix simplifies:

$$\begin{pmatrix} \lambda_1 - \lambda_2 + \alpha & -\eta/2 & 0 \\ -\eta/2 & \alpha & -\partial_x(\lambda_1 - \lambda_2)(t \ln t) \\ 0 & -\partial_x(\lambda_1 - \lambda_2)(t \ln t) & \lambda_1 - \lambda_2 + \alpha - 1 \end{pmatrix}$$

In view of (2.8), this leads us to the matrix

$$(3.9) \quad N^{(1)} := \begin{pmatrix} \alpha_1 & -\eta/2 & 0 \\ -\eta/2 & \alpha_1 & 0 \\ 0 & 0 & \alpha_1 - 1 \end{pmatrix}$$

for the first component and to the matrix

$$(3.10) \quad N^{(2)} := \begin{pmatrix} 2k + \alpha_2 & -\eta/2 & 0 \\ -\eta/2 & \alpha_2 & -2\partial_x k(t \ln t) \\ 0 & -2\partial_x k(t \ln t) & 2k + \alpha_2 - 1 \end{pmatrix}$$

for the second component.

For the matrix $N^{(1)}$ to be positive, it is necessary that $\alpha_1(x) > 1$ for all $x \in U$. However, if $P_* = -k$, then condition (3.3a) in Lemma 3.1 in the finite differentiability case (or the corresponding one in Lemma 3.2 in the C^∞ -case) implies that $\alpha_1(x) < 1$. Hence, in the same way as in Rendall [19], one does not arrive at a well-posedness result for the singular initial value problem yet. However, since the positivity of the energy dissipation matrix is the only part of the hypothesis in Theorem 3.10 of [3] which is violated, we can use instead Theorem 3.12 of [3] to prove well-posedness of the “singular initial value problem with asymptotic solutions of sufficiently high order”.

Let us quickly recapitulate the basics for this singular initial value problem which are discussed in detail in the first Part of this series. Consider any second-order hyperbolic Fuchsian equation of the form (2.5) and, for any given asymptotic data, define

$$\hat{F}[w] := F[w] + t^2 k^2 \partial_x^2 (u_0 + w).$$

Let H be the operator which maps any source function $f_0 = f_0(t, x)$ (having a suitable behavior at $t = 0$) to the remainder $w = v - u_0$, where v the unique solution of the ordinary differential equation

$$D^2 v(t, x) + 2a(x)Dv(t, x) + b(x)v(t, x) = f_0(t, x),$$

consistent with the prescribed asymptotic data. Finally, set $G := H \circ \hat{F}$. As is easily checked, w is the remainder of a solution of the full equations (consistent with the prescribed asymptotic data) if and only if $w = G[w]$, that is, if w is a fixed point of the map G .

Set $w_1 = 0$ and define the sequence

$$w_{j+1} = G[w_j], \quad j = 1, 2, \dots$$

The convergence of this sequence to a fixed point is known for analytic data and for ordinary differential equations, only. Yet, the sequence (w_j) has certain useful properties.

On one hand, the residual of the second-order hyperbolic Fuchsian equation, i.e. the difference between the left- and the right-hand side is of higher order in t (at $t = 0$) if j is larger. Hence, the sequence satisfies the original equations at a higher order of approximation if we choose larger values of j . A disadvantage is that the higher we choose j , the more spatial derivatives of the asymptotic data we need to control. In any case, the main point for the current discussion of well-posedness for the Gowdy equations is the following one.

Define the exponent

$$(3.11) \quad \tilde{\alpha} := \alpha + (j - 2)\kappa\epsilon,$$

where α and ϵ are the quantities introduced above and $\kappa < 1$ is a constant which we can choose arbitrarily. If $v = (P, Q)$ is a solution of the Gowdy equations corresponding to given asymptotic data, we set

$$w := v - u_0 - w_j$$

for some $j \geq 1$. Then it follows from our considerations in the first paper that the equation has a unique solution with remainder $w \in X_{\delta, \tilde{\alpha}, k}$ for some k , provided j is large enough so that the energy dissipation matrix (evaluated with $\tilde{\alpha}$) is positive. Hence, our previous discussion implies that the singular initial value problem with asymptotic solutions of order j is well-posed provided one of the previous lemmas applies. (This is only true if the asymptotic data functions are sufficiently regular.)

We can be more specific about what we mean by j being “sufficiently large”, and we now make some choice for the parameters α_1 , α_2 and ϵ , consistent with Lemma 3.2, which will allow us to

estimate the required size of j . We make no particular effort to choose these quantities optimally, but still the goal is to choose j “reasonably” small. Henceforth, we restrict to the C^∞ -case and $P_*(x) = -k(x)$ with $0 < k(x) < 1$ for all $x \in U$. We introduce positive constants μ_1 and μ_2 (with further restrictions later) and the function $\chi(x) := 1 - 2|x - 1/2|$. The condition (3.8a) states that we must choose $\alpha_1(x)$ and ϵ so that $\alpha_1(x) + \epsilon < \chi(k(x))$. We set

$$(3.12) \quad \alpha_1(x) := 1 - \sqrt{4(k(x) - 1/2)^2 + \mu_1^2},$$

and find $\chi(k(x)) - \alpha_1(x) > \sqrt{1 + \mu_1^2} - 1$ for all $x \in U$, provided $0 < k(x) < 1$. Similarly, we set

$$(3.13) \quad \alpha_2(x) := 1 - \sqrt{4(k(x) - 1/2)^2 + \mu_2^2},$$

and it follows that $\alpha_1(x) - \alpha_2(x) > \sqrt{1 + \mu_2^2} - \sqrt{1 + \mu_1^2}$ for $\mu_2 > \mu_1$. For the conditions (3.8a) and (3.8c) to hold true, we have to choose

$$0 < \mu_1 < \mu_2, \quad \text{and} \quad 0 < \epsilon \leq \min \left(\sqrt{1 + \mu_1^2} - 1, \sqrt{1 + \mu_2^2} - \sqrt{1 + \mu_1^2} \right).$$

Condition (3.8b) is then satisfied automatically.

Now, assume in what follows that $k(x) \in (1/2 - \Delta k, 1/2 + \Delta k)$ for all $x \in U$ for a constant $\Delta k \in (0, 1/2)$. Then it is clear that both functions α_1 and α_2 are positive for all such $k(x)$ if and only if

$$\mu_1 < \mu_2 < \sqrt{1 - 4(\Delta k)^2}.$$

This assumption will be made in the following. In Theorem 3.12 of [3], we could choose j as small as possible if we pick the maximal allowed value for ϵ . Hence, we set

$$\epsilon := \min \left(\sqrt{1 + \mu_1^2} - 1, \sqrt{1 + \mu_2^2} - \sqrt{1 + \mu_1^2} \right).$$

We find easily that

$$\sqrt{1 + \mu_1^2} - 1 \leq \sqrt{1 + \mu_2^2} - \sqrt{1 + \mu_1^2},$$

provided

$$\mu_1^2 \leq \frac{1}{4}(\mu_2^2 + 2\sqrt{1 + \mu_2^2} - 2),$$

and check that this is consistent with the condition $0 < \mu_1 < \mu_2$ made before. In order to make a specific choice, we assume this inequality for μ_1 and hence obtain that

$$(3.14) \quad \epsilon = \sqrt{1 + \mu_1^2} - 1.$$

Now, in order to make the energy dissipation matrix positive, we must choose j so that for all $x \in U$,

$$\begin{aligned} \tilde{\alpha}_1(x) &:= \alpha_1(x) + (j - 2)\kappa\epsilon > 1, \\ \tilde{\alpha}_2(x) &:= \alpha_2(x) + (j - 2)\kappa\epsilon > 1 - 2k(x); \end{aligned}$$

cf. (3.11). These two inequalities are satisfied for all functions k under our assumptions if in particular

$$(3.15) \quad j > 2 + \frac{\sqrt{4(\Delta k)^2 + \mu_2^2}}{\kappa(\sqrt{1 + \mu_1^2} - 1)}.$$

In any case, we choose the maximal value for μ_1

$$(3.16) \quad \mu_1 := \frac{1}{2} \sqrt{\mu_2^2 + 2\sqrt{1 + \mu_2^2} - 2},$$

since this minimizes the value on the right side of (3.15). We find that for this value of μ_1 , the right side of (3.15) is monotonically decreasing in μ_2 and diverges to $+\infty$ for $\mu_2 \rightarrow 0$ for all values of Δk .

Theorem 3.4 (Well-posedness theory for the Gowdy equations). *Consider some asymptotic data*

$$P_* = -k, P_{**}, Q_*, Q_{**} \in C^\infty(U),$$

where k is a smooth function $U \rightarrow (1/2 - \Delta k, 1/2 + \Delta k)$ for a constant $\Delta k \in (0, 1/2)$. Then, for the Gowdy equations with these prescribed data are well-posed in certain weighted Sobolev spaces.

More precisely, the singular initial value problem with asymptotic solutions of order j has a unique solution with remainder $w \in X_{\delta, \alpha + (j-2)\kappa\epsilon, \infty}$ for some sufficiently small $\delta > 0$ and some $\kappa < 1$. Here, the exponents $\alpha = (\alpha_1, \alpha_2)$ and ϵ are given in (3.12), (3.13), and (3.14) explicitly in terms of the data and parameters μ_1, μ_2 chosen such that μ_1 is an explicit expression in μ_2 given in (3.16) while μ_2 is a sufficiently close to (but smaller than) $\sqrt{1 - 4(\Delta k)^2}$, and the order of differentiation j satisfies

$$j > 2 + \frac{2}{\sqrt{3 - 4(\Delta k)^2 + 2\sqrt{2 - 4(\Delta k)^2} - 2}}.$$

The above condition implies that to reach $\Delta k \rightarrow 0$ we need $j > 7$, while $\Delta k \rightarrow 1/2$ requires $j \rightarrow \infty$. Although our estimates may not be quite optimal, the latter implication cannot be avoided.

3.4. Fuchsian analysis for the function Λ . So far we have considered the equations (1.2) for P and Q . We can henceforth assume that these equations are solved identically for all $t > 0$ (and $t \leq \delta$ for some $\delta > 0$) and that hence P and Q are given functions with leading-order behavior (2.9) and remainders in a given $X_{\delta, \alpha, k}$. The equations which remain to be solved in order to obtain a solution of the full Einstein's field equations are (1.3) and (1.4). In particular we are interested in the function Λ in order to obtain the full geometrical information. We must compute Λ also as a singular initial value problem with “data” on the singularity analogously to P and Q . The following discussion resembles the previous one and we only discuss new aspects now.

Clearly, the three remaining equations are overdetermined for Λ and hence solutions will exist only under certain conditions. Let us define the following “constraint quantities” from (1.4)

$$\begin{aligned} C_1(t, x) &:= -\partial_t \Lambda + t(P_x)^2 + e^{2P} t(Q_x)^2 + t(\partial_t P)^2 + e^{2P} t(\partial_t Q)^2, \\ C_2(t, x) &:= -\Lambda_x + 2P_x DP + 2e^{2P} Q_x DQ. \end{aligned}$$

Moreover, we define

$$H(t, x) := -\Lambda_{tt} + \Lambda_{xx} + P_x^2 - P_t^2 + e^{2P}(Q_x^2 - Q_t^2)$$

from (1.3). From the evolution equations for P and Q , we find

$$(3.17) \quad \partial_t C_1 = \partial_x C_2 + H, \quad \partial_t C_2 = \partial_x C_1.$$

These equations have the following consequences. Suppose that we use (1.4b) as an evolution equation for Λ . This implies that $C_1 \equiv 0$ for all $t > 0$. Moreover, suppose that we prescribe data at some $t_0 > 0$ (indeed t_0 is allowed to be zero later) so that $C_2(t_0, x) = 0$ for all $x \in U$. Then the equations imply that $H \equiv 0$ and $C_2 \equiv 0$ for all $t > 0$ and thus we have constructed a solution

of the full set of field equations. Alternatively, let us use (1.3) as the evolution equation for Λ , i.e. $H \equiv 0$. Suppose that we prescribe data so that $C_1(t_0, x) = C_2(t_0, x) = 0$ at some t_0 . It follows that $C_1 \equiv C_2 \equiv 0$ for all $t > 0$ because the evolution system (3.17) for C_1 and C_2 is symmetric hyperbolic. Again, Einstein's field equations are solved.

Now, we want to consider the case $t_0 = 0$. First note that (3.17) is regular even at $t = 0$. Suppose that P and Q are functions with leading-order behavior (2.9) and remainders in a given $X_{\delta, \alpha, k}$ with $k \geq 1$. If there exists a function w_3 so that

$$(3.18) \quad \Lambda(t, x) = \Lambda_*(x) \ln t + \Lambda_{**}(x) + w_3(t, x)$$

with w_3 converging to zero in a suitable norm at $t = 0$ and

$$(3.19) \quad \Lambda_*(x) = k^2(x), \quad \Lambda_{**}(x) = \Lambda_0 + 2 \int_0^x k(\tilde{x}) (-\partial_{\tilde{x}} P_{**}(\tilde{x}) + 2e^{2P_{**}(\tilde{x})} Q_{**}(\tilde{x}) \partial_{\tilde{x}} Q_*(\tilde{x})) d\tilde{x},$$

where Λ_0 is an arbitrary real constant, then

$$\lim_{t \rightarrow t_0} C_2 = 0.$$

Let us first use (1.4b) as an evolution equation for Λ . One can show easily that there exists a unique solution for Λ for $t > 0$ which obeys the two-term expansion above. Our discussion before implies that (1.3) is solved identically for all $t > 0$. Hence we obtain a solution of the full Einstein's field equation. Alternative, choose (1.3) as the evolution equation for Λ now. This equation can be written in second-order hyperbolic Fuchsian form

$$D^2 \Lambda - t^2 \partial_x^2 \Lambda = (t \partial_x P)^2 + (D\Lambda - (DP)^2) + e^{2P} ((t \partial_x Q)^2 - (DQ)^2).$$

Indeed this equation is compatible with the leading-order expansion (3.18) at $t = 0$ and we can show well-posed of this singular initial value problem in the same way as we did for the functions P and Q before using the results of [3]. In particular, for any asymptotic data Λ_* and Λ_{**} , there exists a unique solution of this equation Λ with remainder w_3 in a certain space $X_{\delta, \alpha, k}$. Uniqueness implies that the solution Λ of this equation coincides with the solution for Λ obtained using (1.4b) as the evolution equation. Hence, we have $C_1 \equiv C_2 \equiv 0$ for all $t > 0$, and thus also this method yields a solution of the full Einstein's field equations.

Note that periodicity implies that the asymptotic data for P and Q must satisfy the relation

$$\int_0^{2\pi} k(\tilde{x}) (-\partial_{\tilde{x}} P_{**}(\tilde{x}) + 2e^{2P_{**}(\tilde{x})} Q_{**}(\tilde{x}) \partial_{\tilde{x}} Q_*(\tilde{x})) d\tilde{x} = 0$$

in the case of smooth solutions.

4. NUMERICAL SOLUTIONS OF THE SINGULAR INITIAL VALUE PROBLEM

4.1. The Fuchsian numerical algorithm. We proceed now with the numerical approximation of the singular initial value problem associate with second-order hyperbolic Fuchsian equations. The approximation algorithm proposed now is motivated by our proof of Proposition 3.4 in [3]. For linear source-terms, at least, we have shown that the solution of the singular initial value problem can be approximated by solutions to the *regular initial value problem*. The regular initial value problem is defined by data not at the singular time $t = 0$, but rather at some $t_0 > 0$, and by considering the evolution toward the future (i.e. away from the singular time $t = 0$). Then, letting $t_0 \rightarrow 0$, the sequence of solutions to these regular problems, referred to as *approximate solutions*, converges toward the solution of the singular initial value problem. In [3], we established an explicit error

estimate for these approximate solutions and the result should extend to nonlinear source-terms satisfying a Lipschitz continuity conditions.

The regular initial value problem for second-order hyperbolic equations corresponds to the standard initial value problem for a system of (non-linear) wave equations, and there exists a huge amount of numerical techniques for computing solutions. However, a second-order Fuchsian equation written out with the standard time-derivative ∂_t (instead of D) clearly involves factors $1/t$ or $1/t^2$. Although these are finite for the regular initial value problem, they still can cause severe numerical problems when the initial time t_0 approaches zero, due to the finite representation of numbers in a computer. In order to solve this problem, we introduce a new time coordinate

$$\tau := \ln t,$$

and observe that $D = \partial_\tau$. For instance, the following Euler-Poisson-Darboux equation was already treated in [3]

$$(4.1) \quad \partial_\tau^2 v - \lambda \partial_\tau v - e^{2\tau} \partial_x^2 v = 0,$$

where v is the unknown and λ is assumed to be a non-negative constant. Therefore, there is no singular term in this equation; the main price which we pay, however, is that the singularity $t = 0$ has been “shifted to” $\tau = -\infty$. Another disadvantage is that the characteristic speed of this equation (defined with respect to the τ -coordinate) is e^τ and hence increases exponentially with time. For any explicit discretization scheme, we can thus expect that the CFL-condition will always be violated when we evolve into the future from some time on. We must either adapt the time step to the increasing characteristic speeds, or, when we decide to work with fixed time resolution, accept the fact that (for any given resolution) the numerical solution will eventually become instable. However, this is not expected to be a severe problem, since one can compute the numerical solution with respect to the τ -variable until some finite positive time when the numerical solution is still stable and if necessary switch to a discretization scheme based on the original t -variable afterwards. All the numerical solutions presented in this paper are obtained with respect to the τ -variable without adaption.

We can simplify the following discussion slightly by writing (and implementing numerically) the equation not for the function v but for the remainder $w = v - u_0$ with u_0 being the canonical two-term expansion defined in (3.24) in [3] which is determined by given asymptotic data. According to the proof of Proposition 3.4 in [3], the remainder w of any approximate solution is determined by initial data

$$w(\tau_0, x) = 0, \quad \partial_\tau w(\tau_0, x) = 0, \quad x \in U,$$

for some $\tau_0 \in \mathbb{R}$ successively going to $-\infty$.

Inspired by Kreiss et al. in [17] and by the general idea of the “method of lines”, we proceed as follows to discretize the equation. First we consider second-order Fuchsian ordinary differential equations (written for scalar equations for simplicity)

$$\partial_\tau^2 w + 2a \partial_\tau w + b w = f(\tau),$$

where f is a given function and the coefficients a and b are constants. We discretize the time variable τ so that $\tau_n := \tau_0 + n\Delta\tau$, $w_n := w(\tau_n)$ and $f_n := f(\tau_n)$ for some time step $\Delta\tau > 0$. Then the equation is discretized in second-order accuracy as

$$(4.2) \quad \frac{w_{n+1} - 2w_n + w_{n-1}}{(\Delta\tau)^2} + 2a \frac{w_{n+1} - w_{n-1}}{2\Delta\tau} + b w_n = f_n.$$

Solving this for w_{n+1} allows to compute the solution w at the time τ_{n+1} from the solution at the given and previous time τ_n and τ_{n-1} , respectively. At the initial two time steps τ_0 and τ_1 , we set, consistently with the initial data for w at τ_0 above,

$$(4.3) \quad w_0 = 0, \quad w_1 = \frac{1}{2}(\Delta\tau)^2 f(\tau_0).$$

We will refer to this scheme as the *Fuchsian ODE solver*.

The idea of the “method of lines” for Fuchsian partial differential equations is to discretize also the spatial domain with the spatial grid spacing Δx and to use our Fuchsian ODE solver to integrate one step forward in time at each spatial grid point. The source-term function f , which might now depend on the unknown itself and its first derivatives, is then computed from the data on the current or the previous time levels. Here we understand that spatial derivatives in the source-term are discretized by means of the second-order centered stencil using periodic boundary conditions. A problem is that f , besides spatial derivatives, can also involve time derivatives of the unknown w (in fact this can be the case for Fuchsian ordinary differential equations when the source term depends on the time derivative of the unknown). In order to compute those time derivatives in a systematic manner in second-order accuracy, i.e. without changing the stencil of the Fuchsian ODE solver, we made the following choice. In the code we store the numerical solution not only on two time levels, as it is necessary up to now for the scheme given by (4.2) and (4.3), but on a further third past time level. The time derivatives in the source-term can then be computed from data at the present and previous time steps only as follows

$$\partial_\tau w(\tau_n) = \frac{3w_n - 4w_{n-1} + w_{n-2}}{2\Delta\tau} + O((\Delta\tau)^2).$$

For this, we need to initialize three time levels and hence we set

$$w_2 = 2(\Delta\tau)^2 f(\tau_0),$$

in addition to (4.3).

4.2. Euler-Poisson-Darboux equation. We first tested the Fuchsian ODE solver on Fuchsian ordinary differential equations. However, in the following, we consider the P.D.E.’s set-up directly and present numerical results for the Euler-Poisson-Darboux equation (4.1). The reason for considering this equation is that can be considered as a linear version of each of the Gowdy equations. Recall from [3] that the singular initial value problem with two-term asymptotic data for this equation is well-posed for $0 \leq \lambda < 2$ and becomes ill-posed for $\lambda = 2$. We study now this singular initial value problem numerically for $\lambda > 0$, i.e. we look for solutions

$$v(t, x) = u_*(x) + u_{**}(x)t^\lambda + w(t, x),$$

with remainder w . We choose the asymptotic data $u_* = \cos x$, $u_{**} = 0$. Note that in this case, this leading-order behavior is consistent even with the case $\lambda = 0$. But according to the discussion in [3], it is not consistent with $\lambda = 2$, and we expect that this becomes visible in the numerical solutions. For $u_{**} = 0$ and $0 < \lambda < 2$, we can show that the leading-order behavior of the remainder at $t = 0$ is

$$(4.4) \quad w(t, x) = u_*(x) \left(-\frac{1}{2(2-\lambda)}t^2 + \frac{1}{8(2-\lambda)(4-\lambda)}t^4 + \dots \right).$$

First we check that the numerical solutions converge in second-order when $\Delta\tau$ and Δx are changed proportionally to each other for a given choice of initial time $\tau_0 > 0$. We do not discuss this further here, but eventually we choose the resolution so that discretization errors are negligible

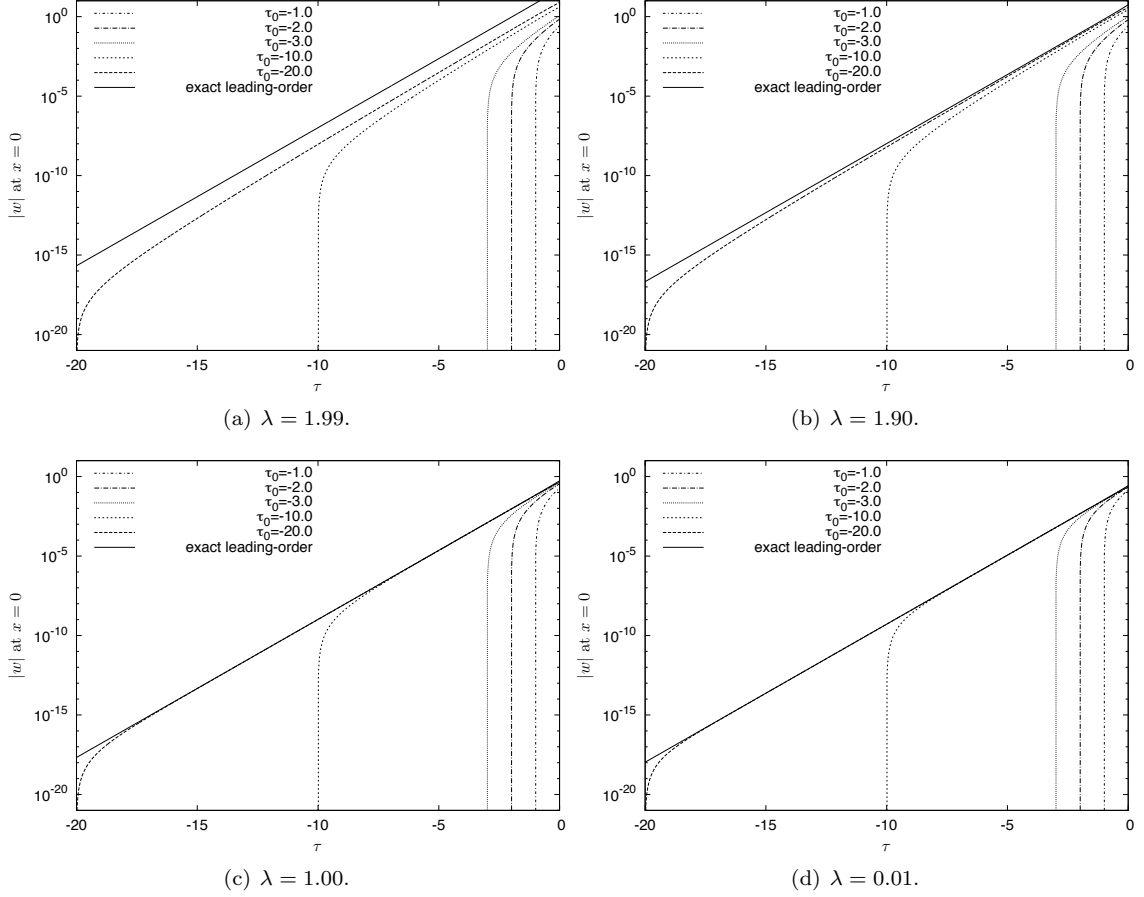


FIGURE 1. Euler-Poisson-Darboux equation.

in the following discussion; the same is true for round-off errors, as we come back to later. In Figure 1, we show the following results obtained with $N = 20$, $\Delta\tau = 0.003$. Here, N is the number of spatial grid points, i.e. one has $\Delta x = 2\pi/N$. We find that for this CFL-parameter $\Delta t/\Delta x \approx 0.01$, the runs are stable until $\tau \approx 5$. For each of the plots of Figure 1, we fix a value of λ and study the convergence of the approximate solutions to the (leading-order of the) exact solution (4.4) for various values of the initial time τ_0 . We plot the value at one spatial point $x = 0$ only. The convergence rate for $\tau_0 \rightarrow -\infty$ is fast if $\lambda = 1$ or $\lambda = 0.01$, but becomes lower, the more λ approaches the value 2, where it becomes zero. This is in exact agreement with our expectations and consistent with the error estimates derived in [3]. Hence the numerical results are very promising.

To close the discussion of this test case, let us add some comment about numerical round-off errors. All numerical runs in this paper were done with double-precision (binary64 of IEEE 754-2008), where the real numbers are accurate for 16 decimal digits. However, for the case $\tau_0 = -20$ for instance, the second spatial derivative of the unknown in the equation is multiplied by $\exp(-40) \approx 10^{-18}$ at the initial time which is not resolved numerically and hence could possibly

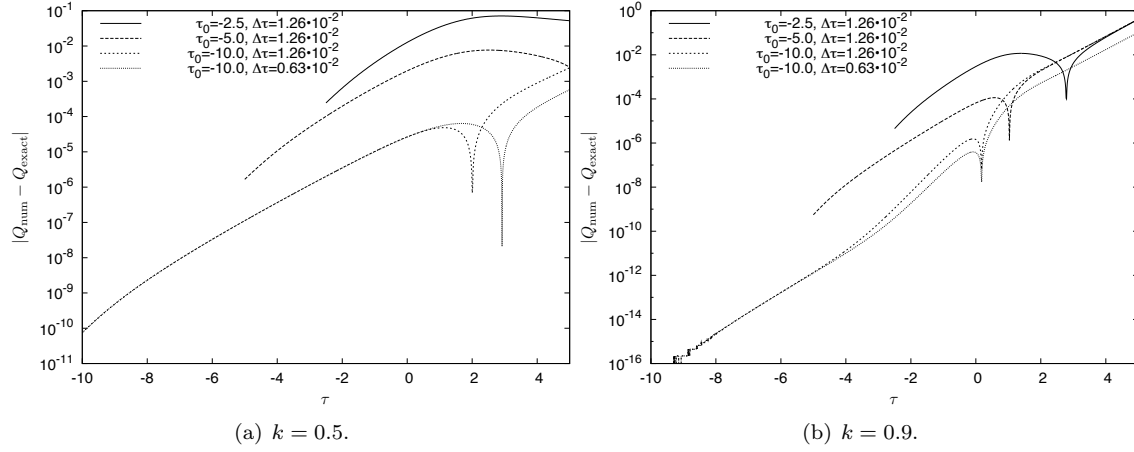


FIGURE 2. Homogeneous pseudo-polarized case (Gowdy equations).

lead to a significant error. This, however, does not seem to be the case since we obtained virtually the same numerical solution with quadruple precision (binary128 of IEEE 754-2008), i.e. when the numbers in the computer are represented with 34 significant decimal digits.

4.3. Singular initial value problem for the Gowdy equations. We continue our discussion with the singular initial value problem for the Gowdy equations. In all of what follows we consider the singular initial value problem with two-term asymptotic data for the Gowdy equations. The fact that this works very well and we get good convergence can be seen as an indication that this singular initial value problem is well-posed. Recall from Theorem 3.4 that our analytical techniques are only sufficient to show that the initial value problem with asymptotic solutions of sufficiently high order is well-posed for the Gowdy equations.

Test 1. Homogeneous pseudo-polarized solutions. Before we proceed with “interesting” solutions of the Gowdy problem, let us start with a test case for which we can construct an explicit solution. Let \tilde{P} and \tilde{Q} be solutions of the polarized equations in the homogeneous case, i.e. one set $\tilde{Q} = 0$ and $\tilde{P}(t, x) = \tilde{P}(t)$. In this case, it follows directly that the exact solution of the Gowdy equations is

$$\tilde{P}(t) = -k \ln t + \tilde{P}_{**},$$

where both k and \tilde{P}_{**} are arbitrary constants. By a reparametrization of the Killing orbits of the form

$$\tilde{x}_2 = x_2/\sqrt{2} + x_3/\sqrt{2}, \quad \tilde{x}_3 = -x_2/\sqrt{2} + x_3/\sqrt{2},$$

where \tilde{x}_2 , and \tilde{x}_3 are the coordinates used to represent the orbits of the polarized solution above, the same solution gets reexpressed in terms of functions

$$(4.5) \quad P = \ln \cosh(-k \ln t + \tilde{P}_{**}), \quad Q = \tanh(-k \ln t + \tilde{P}_{**}).$$

Of course, these functions (P, Q) are again solutions of (1.2). Asymptotically at $t = 0$, they satisfy

$$(4.6) \quad P = -k \ln t + (\tilde{P}_{**} - \ln 2) + \dots, \quad Q = 1 - 2e^{-2\tilde{P}_{**}}t^{2k} + \dots,$$

from which we can read off the corresponding asymptotic data.

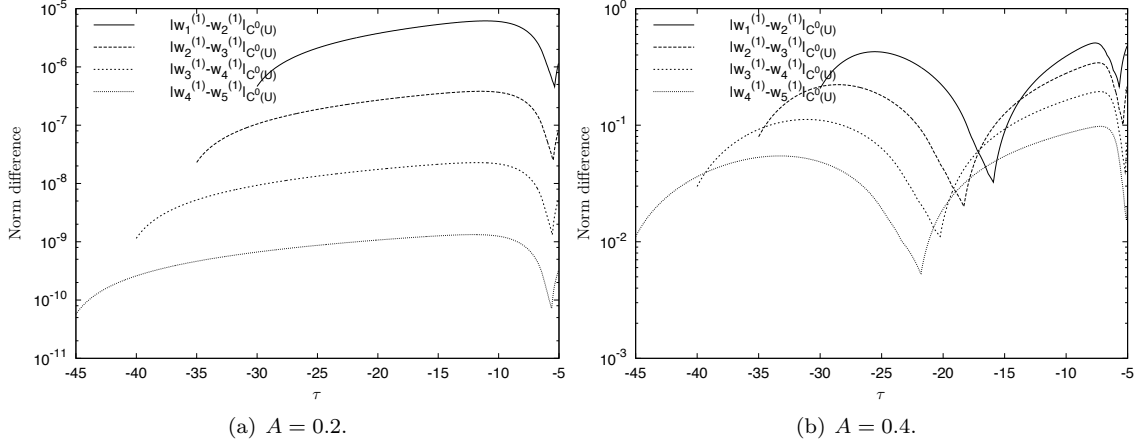


FIGURE 3. Convergence for general Gowdy solutions.

Now we compute the solutions corresponding to these asymptotic data numerically and compare them to the exact solution (4.5). We pick $\tilde{P}_{**} = 1$, so that $P_{**} = 1 - \ln 2$, $Q_* = 1$ and $Q_{**} = -2e^{-2}$. Since the solution is spatially homogeneous – in fact this is an ODE problem – we only need to do the comparison at one spatial point. The results are presented in Figure 2 where we plot the difference of the numerical and the exact value of Q versus time for various values of τ_0 . In the first plot, this is done for $k = 0.5$ and in the second plot for $k = 0.9$. The plots confirm nice convergence of the approximate solutions to the exact solution. The fact that each approximate solution diverges from the exact solution almost exponentially in time is a feature of the approximate solutions themselves and not of the numerical discretization, as is checked by comparing two different values of $\Delta\tau$ in these plots. From our experience with the Euler-Poisson-Darboux equation, we could have expected that the convergence rate is lower in the case $k = 0.9$ than in the case $k = 0.5$ (note that k plays the same role $\lambda/2$). In the case of the Euler-Poisson-Darboux equation, the rate of convergence decreases when λ approaches 2, due to the influence of the second-spatial derivative term in the equation. In the spatially homogeneous case here, however, this term is zero and hence this phenomenon is not present. The “spikes” in Figure 2 are just a consequence of the logarithmic scale of the horizontal axes and the fact that the numerical and exact solutions equal for one instance of time.

Test 2. General Gowdy equations. Now we want to study the convergence for a “generic” inhomogeneous Gowdy case (still ignoring the equation for the quantity Λ). Here we choose the following asymptotic data

$$\begin{aligned} k(x) &= 1/2 + A \cos(x), & Q_* &= 1.0 + \sin(x), \\ P_{**} &= 1 - \ln 2 + \cos(x), & Q_{**} &= -2e^{-2}, \end{aligned}$$

with a constant $A \in (-1/2, 1/2)$. We do not know of an explicit solution in this case. In Figure 3, we show the following numerical results for $A = 0.2$ and $A = 0.4$, respectively. For the given value of A , we compute five approximate solutions numerically with initial times $\tau_0 = -30, -35, -40, -45, -50$ numerically, each with the same resolution $\Delta\tau = 0.01$ and $N = 80$. The resolution parameters have been chosen so that the numerical discretization errors are negligible in the plots of Figure 3. Then, for each time step for $\tau \geq -30$, we compute the supremum norm in space of the difference of the

remainders $w^{(1)}$ of the two approximate solutions given by $\tau_0 = -30$ and $\tau_0 = -35$. In this way we obtain the first curve in each of the plots of Figure 3. The same is done for the difference between the cases $\tau_0 = -35$ and $\tau_0 = -40$ for all $\tau \geq -35$ to obtain the second curve etc. Hence these curves yield a measure of the convergence rate of the approximation scheme, without referring to the exact solution. In agreement with our observation for the Euler-Poisson-Darboux equation, the convergence rate is high if k is close to $1/2$ and becomes lower, the more k touches the “extreme” values $k = 0$ and $k = 1$.

Much in the same way as for the Euler-Poisson-Darboux equation we find that double precision is sufficient for these computations despite of the fact that $\exp(2\tau)$ is 10^{-44} for $\tau = -50$.

4.4. Gowdy spacetimes containing a Cauchy horizon. The papers [9, 10, 11, 14, 15] were devoted to the construction and characterization of Gowdy solutions with Cauchy horizons, in particular in order to prove the strong cosmic censorship conjecture in this class of spacetimes. Spacetimes with Cauchy horizons are expected to have saddle and physically “undesired” properties, in particular they often allow various inequivalent smooth extensions, i.e. the Cauchy problem of Einstein’s field equations does not select one of them uniquely. Some explicit examples are known, but most of the analysis is on the level of existence proofs and asymptotic expansions.

Hence, it is of interest to construct such solutions numerically and analyze them in much greater detail than possible with purely analytic methods. Constructing these solutions numerically, however, is delicate since the strong cosmic censorship conjecture suggests that they are unstable under generic perturbations. It can hence often be expected that numerical errors would most likely “destroy the Cauchy horizon”. This is so, in particular, when the singular time at $t = 0$ is approached backwards in time from some regular Cauchy surface at $t > 0$.

In the Gowdy case, however, there are clear criteria for the asymptotic data so that the corresponding solution of the singular initial value problem has a Cauchy horizon (or only pieces thereof; cf. below) at $t = 0$, as discussed in [9] for the polarized case and in [11] for the general case. Our novel method here allows us to construct such solutions with arbitrary accuracy and it can hence be expected that this allows us to study the saddle properties of such solutions. Our main aim for the following is to compute such a solution and with this demonstrate the feasibility of our approach. A follow-up work will be devoted to the numerical construction and detailed analysis of relevant classes of such solutions.

Motivated by the results in [9], we choose the asymptotic data as follows

$$\begin{aligned} k(x) &= \begin{cases} 1, & x \in [\pi, 2\pi], \\ 1 - e^{-1/x} e^{-1/(\pi-x)}, & x \in (0, \pi), \end{cases} & P_{**}(x) &= 1/2, \\ Q_*(x) &= 0, & Q_{**}(x) &= \begin{cases} 0, & x \in [\pi, 2\pi], \\ e^{-1/x} e^{-1/(\pi-x)}, & x \in (0, \pi), \end{cases} \\ \Lambda_*(x) &= k^2(x), & \Lambda_{**}(x) &= 2. \end{aligned}$$

With these asymptotic data, the corresponding solution has a smooth Cauchy horizon at $(t, x) \in \{0\} \times (\pi, 2\pi)$ (namely where $k \equiv 1$), and a curvature singularity at $(t, x) \in \{0\} \times (0, \pi)$ (namely where $0 < k < 1$). Note that the function k is smooth everywhere (but not analytic). Our analysis in Section 3 shows that we are allowed to set $k = 1$ at some points since $\partial_x Q_* = 0$. This motivates our choice of Q_* . With this, our choice of Q_{**} implies that the solution is polarized on the “domain

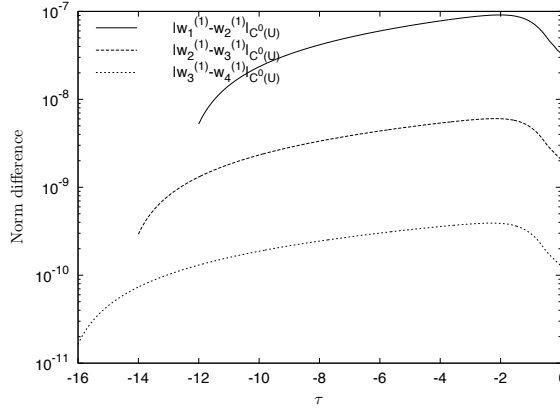


FIGURE 4. Convergence analysis for spacetimes with a Cauchy horizon.

of dependence”² of the “initial data” interval $(\pi, 2\pi)$. All data were chosen as simple as possible to be consistent with the constraints.

First we repeated the same error analysis as for the previous Gowdy case, see Figure 4. For all the runs in the plots, we choose $N = 500$, $\Delta\tau = 0.005$ which guarantees that discretization errors are negligible in the plot. We find that our numerical method allows us to compute the Gowdy solution very accurately. Here, we solve the full system for (P, Q, Λ) .

In Figure 5, we show the numerical solution obtained from $N = 1000$, $\Delta\tau = 0.0025$ and $\tau_0 = -18$. We plot the Kretschmann scalar at two times $\tau = -10$ and $\tau = 0$. Hence, near the time $t = 0$ (corresponding to $\tau = -\infty$), the Kretschmann scalar is large on the spatial interval $(0, \pi)$ while it stays bounded at $(\pi, 2\pi)$. At the later time, the curvature becomes smaller as expected. We also plot the remainders $w^{(1)}$ and $w^{(2)}$ of P and Q , respectively. It is instructive to study how the polarized region inside $(\pi, 2\pi)$ gets “displaced” by the non-polarized solution. We refer the reader to [4] for further investigations, especially a study of past-directed causal geodesics approaching the $t = 0$ -hypersurface near the boundary point $x = \pi$ at the intersection of the Cauchy horizon and the curvature singularity. Furthermore, in [4] we will discuss trapped surfaces in a neighborhood of $t = 0$.

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REFERENCES

- [1] Amorim P., Bernardi C., and LeFloch P.G., Computing Gowdy spacetimes via spectral evolution in future and past directions, *Class. Quantum Grav.* 26 (2009), 1–18.
- [2] Andersson L. and Rendall A.D., Quiescent cosmological singularities, *Commun. Math. Phys.* 218 (2001), 479–511.
- [3] Beyer F. and LeFloch P.G., Second-order hyperbolic Fuchsian systems. General theory, arXiv:1004.4885.

²The notion of “domain of dependence” for the singular initial value problem follows easily from the energy estimate established in [3].

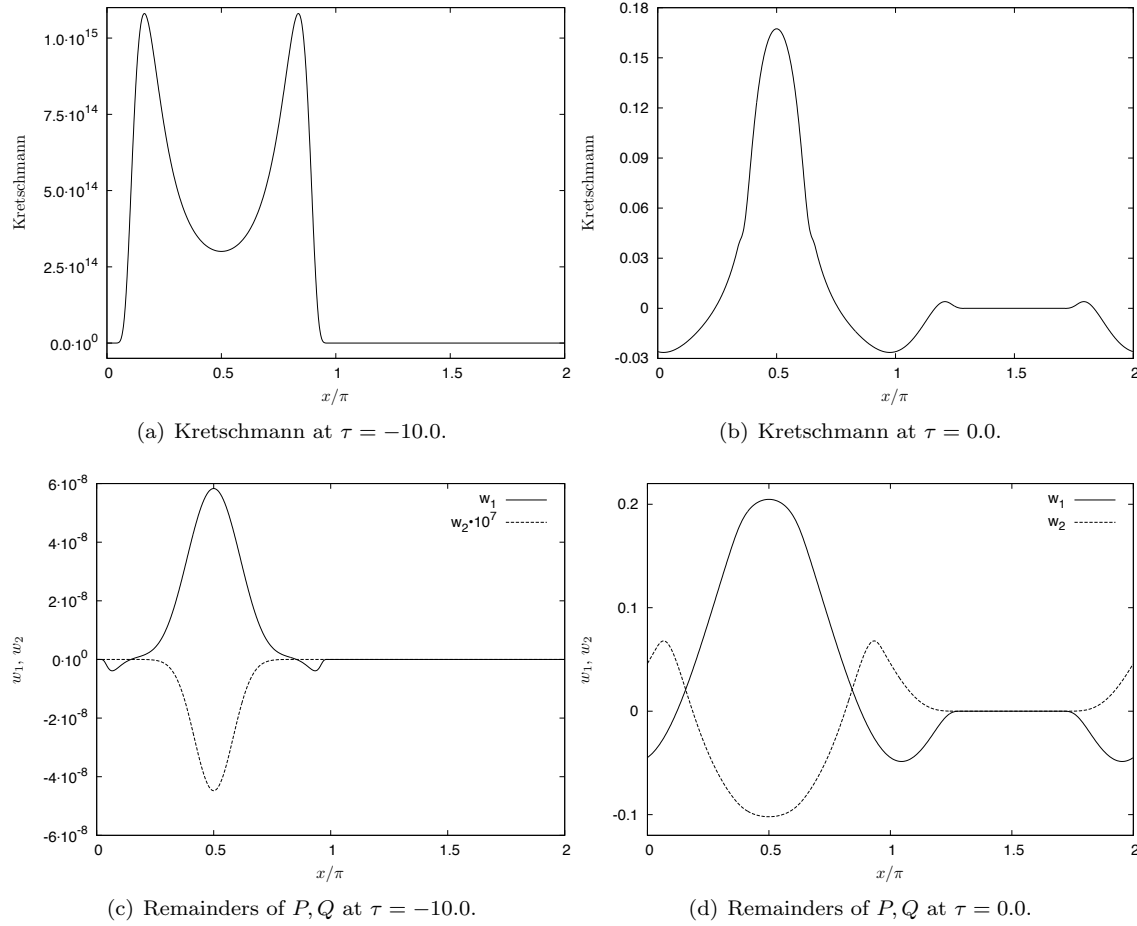


FIGURE 5. Behavior of solutions with a Cauchy horizon.

- [4] Beyer F. and LeFloch P.G., in preparation.
- [5] Choquet-Bruhat Y. and Isenberg J., Half-polarized $U(1)$ symmetric vacuum spacetimes with AVTD behavior, *J. Geom. Phys.* 56 (2006), 1199–1214.
- [6] Choquet-Bruhat Y., Fuchsian partial differential equations, “WASCOM 2007”—14th Conference on Waves and Stability in Continuous Media, World Sci. Publ., Hackensack, NJ, 2008, pp. 153–161.
- [7] Choquet-Bruhat Y., *General relativity and the Einstein equations*, Oxford Math. Monographs, Oxford Univ. Press, Oxford, 2009.
- [8] Chruściel P., On spacetimes with $U(1) \times U(1)$ symmetric compact Cauchy surfaces, *Ann. Phys.* 202 (1990), 100–150.
- [9] Chruściel P., Isenberg J., and Moncrief V., Strong cosmic censorship in polarized Gowdy spacetimes, *Class. Quantum Grav.* 7 (1990), 1671–1680.
- [10] Chruściel P. and Isenberg J., Nonisometric vacuum extensions of vacuum maximal globally hyperbolic spacetimes, *Phys. Rev. D* 48 (1993), 1616–1628.
- [11] Chruściel P. and Lake K., Cauchy horizons in Gowdy spacetimes, *Class. Quantum Grav.* 21 (2004), S153–S169.
- [12] Eardley D. and Moncrief V., The global existence problem and cosmic censorship in general relativity, *Gen. Relat. Grav.* 13 (1981), 887–892.

- [13] Gowdy R.H., Vacuum space-times with two parameter spacelike isometry groups and compact invariant hyper-surfaces: Topologies and boundary conditions, *Ann. Phys.* 83 (1974), 203–241.
- [14] Hennig J. and Ansorg M., Regularity of Cauchy horizons in $S^2 \times S^1$ Gowdy spacetimes, *Class. Quantum Grav.* 27 (2010), 065010.
- [15] Isenberg J. and Moncrief V., Asymptotic behavior of the gravitational field and the nature of singularities in Gowdy spacetimes, *Ann. Phys.* 99 (1990), 84–122.
- [16] Kichenassamy S. and Rendall A.D., Analytic description of singularities in Gowdy spacetimes, *Class. Quantum Grav.* 15 (1998), 1339–1355.
- [17] Kreiss H.O., Petersson N.A., and Ystrom J., Difference approximations for the second-order wave equation, *Siam J. Numer. Anal.* 40 (2002), 1940–1967.
- [18] Moncrief V., Global properties of Gowdy spacetimes with $T^3 \times \mathbb{R}$ topology, *Ann. Phys.* 132 (1981), 87–107.
- [19] Rendall A.D., Fuchsian analysis of singularities in Gowdy spacetimes beyond analyticity, *Class. Quantum Grav.* 17 (2000), 3305–3316.
- [20] Ringström H., Asymptotic expansions close to the singularity in Gowdy spacetimes, *Class. Quantum Grav.* 21 (2004), S305–S322.
- [21] Ringström H., Curvature blow up on a dense subset of the singularity in T³-Gowdy, *J. Hyperbolic Diff. Eqs.* 2 (2005), 547–564.
- [22] Ringström H., Strong cosmic censorship in T³-Gowdy spacetimes, *Ann. of Math.* 170 (2009), 1181–1240.